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Uniqueness and Exponential Decay of Correlations for Some Two-Dimensional Spin Lattice Systems

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We consider a two-dimensional lattice spin system which naturally arises in dynamical systems called coupled map lattice. The configuration space of the spin system is a direct product of mixing subshifts of finite type. The potential is defined on the set of all squares in $\mathbf{Z}^2$ and decays exponentially with the linear size of the square. Via the polymer expansion technique we prove that for sufficiently high temperatures the limit Gibbs distribution is unique and has an exponential decay of correlations.

KEY WORDS: Lattice spin system; shift of finite type; uniqueness of Gibbs state; polymer expansion.

1. INTRODUCTION

The problem considered in this paper arises in the study of ergodic properties of spatially extended dynamical systems called coupled map lattices. These models were introduced in the physical literature by Kaneko[11] as simple examples that demonstrate spatiotemporal chaos.

Let $\mathscr{M}$ be a smooth manifold, f be a map of $\mathscr{M}$ into itself, and $L = \mathbf{Z}^d$ be the integer lattice of dimension d. A popular example of coupled map lattices can be described as follows. The phase space is the direct product of $\mathscr{M}$ over lattice L: $\otimes_L \mathscr{M}$ and the dynamics Φ is a composition of two maps: $\Phi = \otimes_L f \cdot G$, where G is a map on the phase space $\otimes_L \mathscr{M}$ close to the identity map. The map G is usually called the interaction.

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This model was first studied rigorously by Bunimovich and Sinai[2] in the case when $\mathcal{M}$ is a circle, f is an expanding map, and $d=1$. They constructed a natural measure on the phase space that is invariant and mixing under Φ and the spatial translation of the lattice. The idea was to construct Markov partitions for $(\bigotimes_L \mathcal{M}, \Phi)$ and then to represent the dynamics of the couled map lattice by the two-dimensional lattice spin system of statistical mechanics. This technique was later extended by Pesin and Sinai[15] to the situation when f possesses a hyperbolic attractor (see also ref. 9). The symbolic representation in this case becomes $\bigotimes_{\mathbf{Z}} \Sigma_M$, where Σ_M is a subshift of finite type determined by a transfer matrix M and the following diagram is commuting:

$$\begin{array}{ccc} \bigotimes\limits_{\mathbf{Z}} \mathcal{M} & \xrightarrow{(\Phi,\, T)} & \bigotimes\limits_{\mathbf{Z}} \mathcal{M} \\ \Big\uparrow \pi & & \Big\uparrow \pi \\ \bigotimes\limits_{\mathbf{Z}} \Sigma_M & \xrightarrow{\ \tau\ } & \bigotimes\limits_{\mathbf{Z}} \Sigma_M \end{array}$$

where T is the spatial translation, τ is the $\mathbf{Z}^2$-action on $\bigotimes_{\mathbf{Z}} \Sigma_M$ induced by two natural shifts on the lattice $\mathbf{Z}^2$, and π is a semiconjugacy defined by a Markov partition. Thus, the existence, uniqueness, and ergodic properties of invariant measures on $\bigotimes_{\mathbf{Z}} \mathcal{M}$ become problems about Gibbs states for appropriate potentials on $\bigotimes_{\mathbf{Z}} \Sigma_M$.

At first sight, Dobrushin's condition of weak dependence and the corresponding theorems[4, 6, 8, 17] seem to be the most natural tools for the investigation of our model. Unfortunately, Dobrushin's condition requires the total smallness of the potential and is inappropriate for the case when the configuration space is a direct product of the subshifts of finite type. In the latest case the restriction that some pairs of spins cannot be assigned to the neighboring lattice sites corresponds to the infinite potential. Moreover, it is possible to construct examples showing that the Gibbs state is not unique even for infinitely high temperature if the dimension of L is bigger than 1.[3] Some conditions of the uniqueness for the infinite potentials can be found in ref. 7, but they cannot be applied to our situation.

In this paper we study the case of the one-dimensional L which corresponds to the two-dimensional spin lattice system with the configuration space $\bigotimes_{\mathbf{Z}} \Sigma_M$. The Hölder continuous potential functions on the initial coupled map lattice naturally lead to the potential of the spin lattice system which is defined on the set of all squares in $\mathbf{Z}^2$ and decays exponentially with the linear size of the square.[10] Such two-dimensional spin lattice models are the main subject of our study. For them we construct a version of the high-temperature expansion and prove that for sufficiently large temperatures the Gibbs state of the system is unique and has an exponential decay of correlations.

The paper is organized as follows. Section 2 contains all settings and the formulation of the results. In Section 3 we reduce the initial model to a more convenient equivalent model. In Section 4 we construct a polymer expansion for the logarithm of the partition function and Section 5 completes the proof. In Section 6 we briefly discuss possible generalizations of the result. To make the paper self-contained we collected needed results on polymer expansion in the appendix.

2. NOTATIONS AND RESULTS

Let $S=\{1, 2,..., p\}$ be a finite set with p elements and M be a $p\times p$ matrix with entries m_{ij} equal to either 0 or 1. We assume that there exists a positive integer n_0 such that all entries of M^{n_0} are positive. For any volume $V\subseteq \mathbf{Z}^2$ a *configuration* in V is an element $\sigma(V)$ of S^V with the value $\sigma_x(V)$ at point $x=(i,j)\in V$ and $m_{\sigma_{x_1}\sigma_{x_2}}=1$ for any pair $x_1=(i,j)$, $x_2=(i,j+1)\in V$. For the family of configurations $\sigma(V_i)$ in mutually disjoint volumes V_i we denote by $\sum_i \sigma(V_i)$ the corresponding configuration in $\bigcup_i V_i$ provided such a configuration exists. When $V=\mathbf{Z}^2$, we have the configuration space $\Sigma_M^{\mathbf{Z}}=\bigotimes_{\mathbf{Z}} \Sigma_M$, where Σ_M is the subshift generated by the matrix M.

Denote by Q a square from $\mathbf{Z}^2$ of size $l(Q)\times l(Q)$ and consider a poential $U(\sigma(Q))$ defined on the set of all squares in $\mathbf{Z}^2$ and satisfying the condition

$$|U(\sigma(Q))|\leqslant \exp[-l(Q)] \tag{1}$$

Take a finite volume V with a boundary condition $\sigma'(\hat{V})$, $\hat{V}=\mathbf{Z}^2\backslash V$. For any configuration $\sigma(V)$ such that $\sigma(V)+\sigma'(\hat{V})$ is a configuration in $\mathbf{Z}^2$ a conditional Hamiltonian is

$$H(\sigma(V)\mid \sigma'\hat{V}))$$
$$=-\sum_{Q\subseteq V} U(\sigma(Q))-\sum_{Q\cap V\neq\varnothing,\, Q\cap\hat{V}\neq\varnothing} U(\sigma(Q\cap V)+\sigma'(Q\cap\hat{V})) \tag{2}$$

A finite-volume Gibbs distribution is defined by

$$\mu_{V,\sigma'}(\sigma(V))=\frac{\exp[-\beta H(\sigma(V)\mid\sigma'(\hat{V}))]}{\Xi(V\mid\sigma'(\hat{V}))} \tag{3}$$

where the inverse temperature $\beta \geqslant 0$ and

$$\Xi(V \mid \sigma'(\hat{V})) = \sum_{\sigma(V)} \exp[-\beta H(\sigma(V) \mid \sigma'(\hat{V}))] \tag{4}$$

is a partition function in the volume V with the boundary condition $\sigma'(\hat{V})$.

Consider an arbitrary limit Gibbs measure (Gibbs state) corresponding to the Hamiltonian (2). Clearly it can be obtained as the limit of the finite-volume Gibbs distributions μ_{V_m, σ'_m} for an appropriate sequence of boundary conditions $\sigma'_m(\hat{V}_m)$ and $V_m \to \mathbf{Z}^2$ in the van Hove sense. Hence the Gibbs state is unique if for any sequence of boundary conditions $\sigma'_m(\hat{V}_m)$ the limit $\mu = \lim_{m \to \infty} \mu_{V_m, \sigma'_m}$ exists and does not depend on the sequence. We say that μ has an exponential decay of correlations if for any finite $B_1, B_2 \subset \mathbf{Z}^2$ and any configurations $\sigma(B_1)$, $\sigma(B_2)$ there exist an absolute constant r and positive constant $K = K(B_1, B_2)$ such that

$$|\mu(\sigma(B_1) + \sigma(B_2)) - \mu(\sigma(B_1))\, \mu(\sigma(B_2))| \leqslant K \exp[-r \operatorname{dist}(B_1, B_2)]$$

Main Result. For β sufficiently small, model (1)–(4) possesses a unique Gibbs state and this state has an exponential decay of correlations.

Remark 1. Our main result can be easily generalized to the case when instead of a subshift of finite type one has an arbitrary one-dimensional model with the spin taking a finite number of values from the set S and exponentially decaying potential. The sketch of the corresponding proof is given in Section 6.

Remark 2. During the proof we essentially used the fact that our potential is real and it can be transformd into an equivalent nonnegative potential. It should be mentioned also that our method is two-dimensional, as we refer to the fact that lengths of boundaries of two-dimensional volumes are proportional to their diameters. The case of arbitrary dimension is treated by another method in a recent paper.[1]

3. EQUIVALENT MODELS

We show that the model (1)–(4) can be equivalently described by means of more convenient potentials.

Lemma 1. For the potential

$$U'(\sigma(Q)) = U(\sigma(Q)) + \max_{\sigma(Q)} |U(\sigma(Q))| \tag{5}$$

the corresponding Gibbs measure $\mu'_{V,\sigma'}$ coincides with $\mu_{V,\sigma'}$.

Proof. As $U_Q = \max_{\sigma(Q)} |U(\sigma(Q))| \leqslant e^{-l(Q)}$, one has

$$\sum_{Q \cap V \neq \varnothing} U_Q \leqslant \sum_{x \in V} \sum_{Q \ni x} e^{-l(Q)} \leqslant |V| \sum_{l=1}^{\infty} l^2 e^{-l} < \infty \tag{6}$$

where $|V|$ denotes the number of lattice points in V. Hence

$$\begin{aligned}
&\mu'_{V,\sigma'}(\sigma(V)) \\
&= \left\{\exp\left[\beta \sum_{Q \cap V \neq \varnothing} U(\sigma(Q \cap V) + \sigma'(Q \cap \hat{V})) + \beta \sum_{Q \cap V \neq \varnothing} U_Q\right]\right\} \\
&\quad \times \left\{\sum_{\sigma(V)} \exp\left[\beta \sum_{Q \cap V \neq \varnothing} U(\sigma(Q \cap V) + \sigma'(Q \cap \hat{V})) + \beta \sum_{Q \cap V \neq \varnothing} U_Q\right]\right\}^{-1} \\
&= \left\{\exp\left[\beta \sum_{Q \cap V \neq \varnothing} U(\sigma(Q \cap V) + \sigma'(Q \cap \hat{V}))\right]\right\} \\
&\quad \times \left\{\sum_{\sigma(V)} \exp\left[\beta \sum_{Q \cap V \neq \varnothing} U(\sigma(Q \cap V) + \sigma'(Q \cap \hat{V}))\right]\right\}^{-1} \\
&= \mu_{V,\sigma'}(\sigma(V))
\end{aligned} \tag{7}$$

which proves lemma. ∎

In view of Lemma 1 we suppose without loss of generality that

$$U(\sigma(Q)) \geqslant 0 \tag{8}$$

Let us define a beginning $b(Q)$ of a square Q as the leftmost lower corner of Q. Take an integer $L \geqslant n_0$ and consider a rectangle P of size $n(P) \times Ln(P)$ such that its leftmost lower corner $b(P) = (b_1(P), b_2(P))$ has $b_2(P) = rL$, where r and $n(P)$ are integers. We say that the square Q with $b(Q) = (b_1(Q), b_2(Q))$ is associated with the rectangle P if $b_1(Q) = b_1(P)$, $L[b_2(Q)/L] = b_2(P)$, $l(Q) = n(P)$, and hence $Q \subseteq P$ (here $[\cdot]$ denotes the integer part). For any rectangle P we define

$$U(\sigma(P)) = \sum_{Q} U(\sigma(Q)) \tag{9}$$

where the sum is taken over all squares Q associated with the rectangle P. Clearly

$$0 \leqslant U(\sigma(P)) \leqslant L \exp[-n(P)] \tag{10}$$

and absorbing L in β, one can assume that the potential is defined on the rectangles P (instead of squares Q) and satisfies

$$0 \leqslant U(\sigma(P)) \leqslant \exp[-n(P)] \tag{11}$$

Set

$$\partial^{\mathrm{I}} V = \{x \in V \mid \mathrm{dist}(x, \hat{V}) = 1\}, \qquad \partial^{\mathrm{E}} V = \{x \in \hat{V} \mid \mathrm{dist}(x, V) = 1\}$$

We call $\partial^{\mathrm{I}} V$ and $\partial^{\mathrm{E}} V$ the internal and external boundaries of V, respectively. Observe that every finite volume V can be uniquely partitioned into vertical segments V_n, each segment being a connected component of the intersection of V and some vertical line. The points of $\partial^{\mathrm{E}} V$ adjacent to V_n from above and from below we denote by $a(V_n)$ and $b(V_n)$, respectively. The collection of such elements will be denoted by $a(V)$ and $b(V)$. In addition we restrict our considerations to the volumes with

$$L[a(V_n)/L] = a(V_n) \qquad \text{and} \qquad L[b(V_n) + 1/L] - 1 = b(V_n)$$

As we still allow arbitrary boundary conditions, it is enough to prove the uniqueness of the limit Gibbs state when the limit is taken only over growing to $\mathbf{Z}^2$ volumes of the special shape described above.

4. POLYMER EXPANSION OF THE LOGARITHM OF THE PARTITION FUNCTION

Theorem. Suppose that $U(\sigma(P))$ is a poential defined on rectangles of size $n(P) \times Ln(P)$ and satisfying (11). Then there exists an absolute constant β_0 such that for $\beta \leqslant \beta_0$ and any finite volume V of the type described in Section 3 and arbitrary boundary condition $\sigma'(\hat{V})$, the logarithm of the partition function $\Xi(V \mid \sigma'(\hat{V}))$ admits an absolutely convergent polymer expansion of the form (A.4) in the appendix. As a consequence the model defined by potential $U(\sigma(P))$ possesses, for $\beta \leqslant \beta_0$, a unique Gibbs state with the exponential decay of correlations.

To prove this theorem we first define contours and corresponding polymers and then verify the condition of general Theorem A.1 (see appendix).

Evidently in (3) the numerator and the denominator can be multiplied by any nonnegative normalizing factor without changing the value of $\mu_{V,\sigma'}(\sigma(V))$. To choose this factor in a proper way we partition the finite volume V into vertical segments V_n and denote the distance between $a(V_n)$

and $b(V_n)$ by $\|V_n\| = |V_n| + 1$. The number of configurations in V with boundary condition $\sigma'(\hat{V})$ can be calculated as

$$N(V \mid \sigma'(\partial^{\mathrm{E}} V)) = \prod_n N(V_n \mid \sigma'_{a(V_n)}, \sigma'_{b(V_n)}) \tag{12}$$

where $N(V_n \mid \sigma'_{a(V_n)}, \sigma'_{b(V_n)})$ is the matrix element of $M^{\|V_n\|}$ indexed by $\sigma'_{a(V_n)}$, $\sigma'_{b(V_n)}$. By the Perron–Frobenius theorem both M and the adjoint matrix M^* have a unique maximal eigenvalue $\lambda > 1$ and the corresponding eigenvectors $\mathbf{e}$ and $\mathbf{e}^*$ with positive components e_σ and e^*_σ. We normalize $\mathbf{e}$ and $\mathbf{e}^*$ in such a way that $\sum_\sigma e_\sigma e^*_\sigma = 1$. Using the Jordan normal form for the matrix M, one can show that

$$\begin{aligned} &N(V_n \mid \sigma'_{a(V_n)}, \sigma'_{b(V_n)}) \\ &\quad = e_{\sigma'_{a(V_n)}} e^*_{\sigma'_{b(V_n)}} \lambda^{\|V_n\|} [1 + F(V_n \mid \sigma'_{a(V_n)}, \sigma'_{b(V_n)})] \end{aligned} \tag{13}$$

where for some $0 < \rho(M) < 1$ and $\nu(M) > 0$

$$|F(V_n \mid \sigma'_{a(V_n)}, \sigma'_{b(V_n)})| \leqslant \nu(M)\, \rho(M)^{\|V_n\|} \tag{14}$$

Now we define

$$\begin{aligned} L(V) &= \lambda^{-\sum_n \|V_n\|} \\ E(\sigma(\partial^{\mathrm{E}} V)) &= \left(\prod_n e_{\sigma_{a(V_n)}}\right)^{-1} \left(\prod_n e^*_{\sigma_{b(V_n)}}\right)^{-1} \\ E^*(\sigma(\partial^{\mathrm{E}} V)) &= \left(\prod_n e^*_{\sigma_{a(V_n)}}\right)^{-1} \left(\prod_n e_{\sigma_{b(V_n)}}\right)^{-1} \end{aligned} \tag{15}$$

Similarly, we define $E(\sigma(\partial^{\mathrm{I}} V))$ and $E^*(\sigma(\partial^{\mathrm{I}} V))$ by using the top and bottom elements of V_n instead of $a(V_n)$ and $b(V_n)$. We equivalently redefine a finite-volume Gibbs distribution as

$$\mu_{V,\sigma'}(\sigma(V)) = \frac{L(V)\, E(\sigma'(\partial^{\mathrm{E}} V))}{\Xi(V \mid \sigma'(\hat{V}))} \exp[-\beta H(\sigma(V) \mid \sigma'(\hat{V}))] \tag{16}$$

where

$$\Xi(V \mid \sigma'(\hat{V})) = L(V)\, E(\sigma'(\partial^{\mathrm{E}} V)) \sum_{\sigma(V)} \exp[-\beta H(\sigma(V) \mid \sigma'(\hat{V}))] \tag{17}$$

and

$$H(\sigma(V) \mid \sigma'(\hat{V})) = -\sum_{P \cap V \neq \varnothing} U(\sigma(P \cap V) + \sigma'(P \cap \hat{V})) \tag{18}$$

Our nearest aim is to obtain for $\Xi(V \mid \sigma'(\hat{V}))$ a contour representation of type (A.1) in the appendix. Consider a finite family of rectangles $\Gamma = \{P_i\}$ such that $\bar{\Gamma} = \bigcup_i P_i$ is a connected subset of $\mathbf{Z}^2$. The following algorithm produces a minimal covering $\gamma(\Gamma)$ of $\bar{\Gamma}$:

(i) Fix the *leftmost lower* point in $\bar{\Gamma}$ and from all rectangles of Γ which begin at this point include in $\gamma(\Gamma)$ the rectangle P_{i_1} with the maximal linear size $n(P_{i_1})$.

(ii) Suppose that the rectangles $P_{i_1},..., P_{i_k}$ are already selected to $\gamma(\Gamma)$ during the previous steps of the algorithm and fix the *leftmost lower* point $x \in \bar{\Gamma} \setminus (\bigcup_{j=1}^{k} P_{i_j})$. Consider all rectangles of Γ covering x. Among them choose the rectangles with the maximal right upper corner (here maximal means *rightmost upper*). From the latter family of restangles include in $\gamma(\Gamma)$ the rectangle $P_{i_{k+1}}$ which has the maximal linear size.

(iii) Repeat step (ii) until $\bar{\Gamma}$ is totally covered, i.e., $\bar{\Gamma} = \bigcup_j P_{i_j}$.

Clearly the algorithm results in a unique family $\gamma(\Gamma) = \{P_{i_j}\}$ which is called the precontour of Γ and has the property that for every $P_{i_j} \in \gamma(\Gamma)$ there exists a point x covered by P_{i_j} but not belonging to other rectangles of $\gamma(\Gamma)$.

Now we define a *precontour* $\gamma = \{P_j\}$ (not related to any Γ) as a family of rectangles such that $\bar{\gamma} = \bigcup_j P_j$ is a connected subset of $\mathbf{Z}^2$ and every P_j contains a point which does not belong to any other rectangle of γ.

We say that a rectangle P is *compatible* with precontour $\gamma = \{P_j\}$ and denote it $P \prec \gamma$ if for $\Gamma = \{P_j\} \cup P$ one has $\gamma(\Gamma) = \gamma$. Obviously any $P \prec \gamma$ belongs to $\bar{\gamma}$ and any P embedded into some $P_j \in \gamma$ is compatible with γ. It is also clear that some of the rectangles $P \subseteq \bar{\gamma}$ can be incompatible with γ.

For any rectangle P and any configuration $\sigma(P)$ introduce

$$U(\beta, \sigma(P)) = \exp[\beta U(\sigma(P))] - 1 \tag{19}$$

Obviously

$$0 \leqslant U(\beta, \sigma(P)) \leqslant 2\beta e^{-n(P)} \tag{20}$$

for β small enough and one has

$$\Xi(V \mid \sigma'(\hat{V})) = L(V)\, E(\sigma'(\partial^{\mathrm{E}} V)) \times \sum_{\sigma(V)} \prod_{P:\, P \cap V \neq \varnothing} (1 + U(\beta, \sigma(P \cap V) + \sigma'(P \cap \hat{V}))) \tag{21}$$

Opening all brackets in the product, one gets

$$\Xi(V \mid \sigma'(\hat{V})) = L(V)\, E(\sigma'(\partial^{\mathrm{E}} V)) \times \sum_{\sigma(V)} \prod_{\{P_i\}:\, P_i \cap V \neq \varnothing} \prod_{P:\, P \in \{P_i\}} U(\beta, \sigma(P \cap V) + \sigma'(P \cap \hat{V})) \tag{22}$$

Regrouping terms in (22), we obtain

$$\Xi(V \mid \sigma'(\hat{V})) = L(V)\, E(\sigma'(\partial^{\mathrm{E}} V)) \times \sum_{\sigma(V)} \sum_{\{\gamma_i\}^{\mathrm{ext}} \cap V \neq \varnothing} \prod_i \Bigg(\prod_{P \in \gamma_i} U(\beta, \sigma(P \cap V) + \sigma'(P \cap \hat{V})) \times \prod_{P \prec \gamma_i} (1 + U(\beta, \sigma(P \cap V) + \sigma'(P \cap \hat{V})) \Bigg) \tag{23}$$

where the second sum is taken over all collections of precontours $\{\gamma_i\}^{\mathrm{ext}}$ having:

(i) $\operatorname{dist}(\bar{\gamma}_{i_1}, \bar{\gamma}_{i_2}) > 1$ for any $\gamma_{i_1}, \gamma_{i_2} \in \{\gamma_i\}^{\mathrm{ext}}$.

(ii) $P \cap V \neq \varnothing$ for any $P \in \gamma_i$ and any $\gamma_i \in \{\gamma_i\}^{\mathrm{ext}}$.

Such collections are called collections of *mutually external precontours* [condition (i)] having nonempty intersection with volume V [condition (ii)]. An empty collection is also counted in the partition function (23) with all products being identically 1.

Given a precontour γ with $\bar{\gamma} \cap V \neq \varnothing$ and fixed configuration $\sigma(\partial^{\mathrm{I}} \bar{\gamma} \cap V)$, we define a *precontour partition function*

$$\begin{aligned} &\Xi(\gamma, \sigma^{\mathrm{I}} \bar{\gamma} \cap V) \mid \sigma'(\hat{V})) \\ &\quad = L((\bar{\gamma} \backslash \partial^{\mathrm{I}} \bar{\gamma}) \cap V)\, E^*(\sigma(\partial^{\mathrm{I}} \bar{\gamma} \cap V))^{-1}\, E(\sigma'(\partial^{\mathrm{E}} V \cap \bar{\gamma})) \\ &\quad\quad \times \sum_{\sigma((\bar{\gamma} \backslash \partial^{\mathrm{I}} \bar{\gamma}) \cap V)} \prod_{P \in \gamma} U(\beta, \sigma(P \cap V) + \sigma'(P \cap \hat{V})) \\ &\quad\quad \times \prod_{P \prec \gamma} (1 + U(\beta, \sigma(P \cap V) + \sigma'(P \cap \hat{V}))) \end{aligned} \tag{24}$$

For the finite volume V with fixed configuration $\sigma'(\partial^{\mathrm{E}} V)$ on its external boundary we set

$$Z(V \mid \sigma'(\partial^{\mathrm{E}} V)) = L(V)\, E(\sigma'(\partial^{\mathrm{E}} V))\, N(V \mid \sigma'(\partial^{\mathrm{E}} V)) \tag{25}$$

Then the partition function (23) becomes

$$\begin{aligned}
&\Xi(V \mid \sigma'(\hat{V})) \\
&= \sum_{\{\gamma_i\}^{\text{ext}} \cap V \neq \varnothing} \; \sum_{\sigma((\bigcup_i \partial^{\mathrm{I}} \bar{\gamma}_i) \cap V)} Z\left(V \Big\backslash \left(\bigcup_i \bar{\gamma}_i \right) \Big| \, \sigma' \left(\partial^{\mathrm{E}} V \Big\backslash \left(\bigcup_i \bar{\gamma}_i \right) \right) \right. \\
&\quad \left. + \sum_i \sigma(\partial^{\mathrm{I}} \bar{\gamma}_i \cap V) \right) \prod_i \Xi(\gamma_i, \sigma(\partial^{\mathrm{I}} \bar{\gamma}_i \cap V) \mid \sigma'(\hat{V})) \qquad (26)
\end{aligned}$$

On the other hand, one can define

$$\begin{aligned}
&\Xi^*(V \mid \sigma'(\partial^{\mathrm{E}} V)) \\
&= L(V)\, E(\sigma'(\partial^{\mathrm{E}} V)) \sum_{\sigma(V)} \prod_{P:\, P \subseteq V} (1 + U(\beta, \sigma(P))) \\
&= L(V)\, E(\sigma'(\partial^{\mathrm{E}} V)) \\
&\quad \times \sum_{\sigma(V)} \sum_{\{\gamma_i\}^{\text{ext}} \subseteq V} \prod_i \left(\prod_{P \in \gamma_i} U(\beta, \sigma(P)) \prod_{P \prec \gamma_i} (1 + U(\beta, \sigma(P))) \right) \qquad (27)
\end{aligned}$$

where $\{\gamma_i\}^{\text{ext}} \subseteq V$ means that $P \subseteq V$ for every $P \in \gamma_i$ and every $\gamma_i \in \{\gamma_i\}^{\text{ext}}$. For the precontours γ contributing to $\sum_{\{\gamma_i\}^{\text{ext}} \subseteq V}$ its partition function, i.e., expression (24), can be simplified to

$$\begin{aligned}
\Xi(\gamma, \sigma(\partial^{\mathrm{I}} \bar{\gamma})) &= \frac{L(\bar{\gamma} \backslash \partial^{\mathrm{I}} \bar{\gamma})}{E^*(\sigma(\partial^{\mathrm{I}} \bar{\gamma}))} \\
&\quad \times \sum_{\sigma(\bar{\gamma} \backslash \partial^{\mathrm{I}} \bar{\gamma})} \prod_{P \in \gamma} U(\beta, \sigma(P)) \prod_{P \prec \gamma} (1 + U(\beta, \sigma(P)) \qquad (28)
\end{aligned}$$

as there is no dependence on $\sigma'(\hat{V})$. So an analog of expression (26) reads

$$\begin{aligned}
&\Xi^*(V \mid \sigma'(\partial^{\mathrm{E}} V)) \\
&= \sum_{\{\gamma_i\}^{\text{ext}} \subseteq V} \; \sum_{\sigma(\bigcup_i \partial^{\mathrm{I}} \bar{\gamma}_i)} Z\left(V \Big\backslash \left(\bigcup_i \bar{\gamma}_i \right) \Big| \, \sigma'(\partial^{\mathrm{E}} V) + \sigma\left(\bigcup_i \partial^{\mathrm{I}} \bar{\gamma}_i \right) \right) \\
&\quad \times \prod_i \Xi(\gamma_i, \sigma(\partial^{\mathrm{I}} \bar{\gamma}_i)) \\
&= \sum_{\{\gamma_i\}^{\text{ext}} \subseteq V} \; \sum_{\sigma(\bigcup_i \partial^{I} \bar{\gamma}_i)} Z\left(V \Big\backslash \left(\bigcup_i \bar{\gamma}_i \right) \Big| \, \sigma'(\partial^{\mathrm{E}} V) + \sigma\left(\bigcup_i \partial^{\mathrm{I}} \bar{\gamma}_i \right) \right) \\
&\quad \times \prod_i W(\gamma_i, \sigma(\partial^{\mathrm{I}} \bar{\gamma}_i))\, \Xi^*(\bar{\gamma}_i \backslash \partial^{\mathrm{I}} \bar{\gamma}_i \mid \sigma(\partial^{\mathrm{I}} \bar{\gamma}_i)) \qquad (29)
\end{aligned}$$

where the *statistical weight* of the precontour not intersecting $\hat{V}$ is defined by

$$W(\gamma, \sigma(\partial^{\mathrm{I}}\bar{\gamma})) = \frac{\Xi(\gamma, \sigma(\partial^{\mathrm{I}}\bar{\gamma}))}{\Xi^*(\bar{\gamma}\backslash\partial^{\mathrm{I}}\bar{\gamma} \mid \sigma(\partial^{\mathrm{I}}\bar{\gamma}))} \tag{30}$$

Iterating above expression (30), one obtains

$$\begin{aligned}
&\Xi^*(V \mid \sigma'(\partial^{\mathrm{E}} V)) \\
&\quad = \sum_{\{\gamma_i\} \subseteq V} \sum_{\sigma(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i)} Z\left(V\Big\backslash\left(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i\right)\Big|\, \sigma'(\partial^{\mathrm{E}} V) + \sigma\left(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i\right)\right) \\
&\qquad \times \prod_i W(\gamma_i, \sigma(\partial^{\mathrm{I}}\bar{\gamma}_i))
\end{aligned} \tag{31}$$

where the external sum is now taken over all *compatible collections of precontours* belonging to volume V. A collection $\{\gamma_i\}$ is called a compatible collection of precontours if for any two $\gamma_{i_1}, \gamma_{i_2} \in \{\gamma_i\}$ either $\mathrm{dist}(\bar{\gamma}_{i_1}, \bar{\gamma}_{i_2}) > 1$ or $\bar{\gamma}_{i_1} \subseteq \bar{\gamma}_{i_2}\backslash\partial^{\mathrm{I}}\bar{\gamma}_{i_2}$. Similarly,

$$\begin{aligned}
&\Xi(V \mid \sigma'(\hat{V})) \\
&\quad = \sum_{\{\gamma_i\} \cap V \neq \varnothing} \sum_{\sigma((\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i) \cap V)} Z\left(V\Big\backslash\left(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i\right)\Big|\, \sigma'\left(\partial^{\mathrm{E}} V\Big\backslash\left(\bigcup_i \bar{\gamma}_i\right)\right)\right. \\
&\qquad \left. + \sum_i \sigma(\partial^{\mathrm{I}}\bar{\gamma}_i \cap V)\right) \prod_i W(\gamma_i, \sigma(\partial^{\mathrm{I}}\bar{\gamma}_i \cap V) \mid \sigma'(\hat{V}))
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
&W(\gamma, \sigma(\partial^{\mathrm{I}}\bar{\gamma} \cap V) \mid \sigma'(\hat{V})) \\
&\quad = \frac{\Xi(\gamma, \sigma(\partial^{\mathrm{I}}\bar{\gamma} \cap V) \mid \sigma'(\hat{V}))}{\Xi^*((\bar{\gamma} \cap V)\backslash\partial^{\mathrm{I}}\bar{\gamma} \mid \sigma(\partial^{\mathrm{I}}\bar{\gamma} \cap V) + \sigma'(\partial^{\mathrm{E}} V \cap \bar{\gamma}))}
\end{aligned} \tag{33}$$

is a general expression for the statistical weight of precontour which coincides with (30) for the precontours nonintersecting $\hat{V}$.

Represent pictorially the precontour γ as $\partial^{\mathrm{I}}\bar{\gamma}$ drawn in the form of closed broken line (geometrical contour). Then the external sum in (32) extends over all collections of nonintersecting geometrical contours possibly with one geometrical contour embedded into the interior of another geometrical contour. Provided the configuration σ is fixed in the union of all geometrical contours, the factor $Z(\cdot \mid \cdot)$ in (32) counts the normalized number of configurations in the complement (to V) of the union of all geometrical contours. As the precontour γ is not only the boundary $\partial^{\mathrm{I}}\bar{\gamma}$, but the entire family of rectangles, the picture above is useful but incomplete.

Now we derive a convenient representation for $Z(\cdot|\cdot)$. Let $\{T_n\}$ be a partition of $V\backslash(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i)$ into vertical segments. In view of (12), (13), and (15)

$$Z\left(V\Big\backslash\Big(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i\Big)\,\Big|\,\sigma'\Big(\partial^{\mathrm{E}}V\Big\backslash\Big(\bigcup_i \bar{\gamma}_i\Big)\Big)+\sum_i \sigma(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V)\right)$$
$$=\prod_n \frac{N(T_n\mid \sigma''_{a(T_n)},\sigma''_{b(T_n)})}{e_{\sigma''_{a(T_n)}}e^*_{\sigma''_{b(T_n)}}\lambda^{\|T_n\|}}=\prod_n (1+F(T_n\mid \sigma''_{a(T_n)},\sigma''_{b(T_n)}))$$
$$=\sum_{\{T_j\}}\prod_j F(T_j\mid \sigma''_{a(T_j)},\sigma''_{b(T_j)}) \tag{34}$$

where

$$\sigma''=\sigma'\Big(\partial^{\mathrm{E}}V\Big\backslash\Big(\bigcup_i \bar{\gamma}_i\Big)\Big)+\sum_i \sigma(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V)$$

and the sum $\sum_{\{T_j\}}$ is taken over all possible collections (including the empty collection) of vertical segments from the partition of $V\backslash(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}^i)$. If one substitutes the right-hand side of (34) in (32), then the resulting expression can be written in terms of contours which are defined below.

A *contour* is a triple $\Omega=(\{\gamma_i\},\{\tau_j\},\sigma)$ such that:

(a) $\{\gamma_i\}\cap V\neq\varnothing$ is a compatible collection of precontours.

(b) $\{\tau_j\}\subseteq V\backslash(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}^i)$ is a collection of mutually disjoint finite vertical segments with $a(\tau_j)$, $b(\tau_j)\in\bigcup_j(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V)\cup\partial^{\mathrm{E}}V$.

(c) σ is configuration in $\bigcup_i(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V)$.

(d) Either $\{\gamma_i\}$ is nonempty and for every τ_j at least one of its ends $[a(\tau_j)$ or $b(\tau_j)]$ belongs to $\bigcup_i(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V)$ or $\{\gamma_i\}$ is empty and $\{\tau_j\}$ consists of a single segment τ with $a(\tau)$, $b(\tau)\in\partial^{\mathrm{E}}V$.

(e) For every pair $\gamma_{i'}$ and $\gamma_{i''}$ there exists a sequence

$$\gamma_{i'}=\gamma_{i_1},\tau_{j_1},\ldots,\gamma_{i_s},\tau_{j_s},\gamma_{i_{s+1}}=\gamma_{i''}$$

such that for any $1\leqslant k\leqslant s$ either $a(\tau_{j_k})\in\partial^{\mathrm{I}}\bar{\gamma}_{i_k}$ and $b(\tau_{j_k})\in\partial^{\mathrm{I}}\bar{\gamma}_{i_{k+1}}$ or $b(\tau_{j_k})\in\partial^{\mathrm{I}}\bar{\gamma}_{i_k}$ and $a(\tau_{j_k})\in\partial^{\mathrm{I}}\bar{\gamma}_{i_{k+1}}$.

The definition above is clearly V dependent. In the special case $V=\mathbf{Z}^2$ we obtain so-called *free* contours.

It is useful to represent pictorially a contour as a family of pairwise nonintersecting geometrical contours joined in the connected structure by means of some number of vertical segments. Some of the geometrical contours can be embedded into other geometrical contours and some of the segments can join the whole structure with $\partial^{\mathrm{E}}V$.

Given a contour $\Omega=(\{\gamma_i\}, \{\tau_j\}, \sigma)$, we define $\bar{\Omega}^\tau=\bigcup_j \tau_j$, $\bar{\Omega}^\gamma=\bigcup_i \bar{\gamma}_i$, $\bar{\Omega}=\bar{\Omega}^\tau \cup \bar{\Omega}^\gamma$, $\tilde{\Omega}=\bar{\Omega}^\tau \cup (\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}^i)$, and

$$W(\Omega \mid \sigma'(\hat{V}))=\prod_i W(\gamma_i, \sigma(\partial^{\mathrm{I}}\bar{\gamma}_i \cap V) \mid \sigma'(\hat{V})) \prod_j F(\tau_j \mid \sigma''_{a(\tau_j)}, \sigma''_{b(\tau_j)}) \quad (35)$$

where $\sigma''=\sigma(\bigcup_i (\partial^{\mathrm{I}}\bar{\gamma}_i \cap V))+\sigma'(\partial^{\mathrm{E}} V \backslash (\bigcup_i \bar{\gamma}_i))$. This leads to the representations

$$\Xi(V \mid \sigma'(\hat{V}))=\sum_{\{\Omega_l\} \cap V \neq \varnothing} \prod_l W(\Omega_l \mid \sigma'(\hat{V})) \quad (36)$$

and

$$\Xi^*(V \mid \sigma'(\partial^{\mathrm{E}} V))=\sum_{\{\Omega_l\} \subseteq V} \prod_l W(\Omega_l \mid \sigma'(\hat{V})) \quad (37)$$

The sums in (36) and (37) extend over *compatible collections of contours* having nonempty intersection with V and belonging to V, respectively. A contour Ω belongs to the volume V if the corresponding precontours $\gamma_i \subseteq V$ and $\bar{\Omega} \subseteq V$. A contour Ω has nonempty intersection with volume V if $\{\gamma_i\} \cap V \neq \varnothing$ and $\bar{\Omega}^\tau \subseteq V$. A collection $\{\Omega_l\}$ is *compatible* if for any Ω_{l_1} and Ω_{l_2} one has $\tilde{\Omega}_{l_1} \cap \tilde{\Omega}_{l_2}=\varnothing$ and the total collection $\{\gamma_i(\Omega_{l_1}), \gamma_i(\Omega_{l_2})\}$ is a compatible collection of precontours. For free contours we use the notation $W(\Omega_l)$ instead of $W(\Omega_l \mid \sigma'(\hat{V}))$, as in this case the right-hand side of (35) does not depend on $V=\mathbf{Z}^2$.

By construction (the shape of P and V) $\|\tau_j\|=n(\tau_j) L$ with $n(\tau_j)$ being a positive integer. Choosing L large enough, one can make $\delta=\nu(M)\, \rho(M)^L$ arbitrarily small and we treat δ as the second small parameter of our calculations (the first one is β).

Lemma 2. The statistical weight of contour $\Omega=(\{\gamma_i\}, \{\tau_j\}, \sigma)$ satisfies

$$|W(\Omega_l \mid \sigma'(\hat{V}))| \leqslant \frac{\prod_j \delta^{n(\tau_j)} \prod_i \prod_{P \in \gamma_i} 2\beta e^{-n(P)+\beta^{1/2} n(P)}}{E(\sigma((\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}^i) \cap V))\, E^*(\sigma((\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i) \cap V))} \quad (38)$$

Proof. Clearly $\prod_j \nu(M)\, \rho(M)^{\|\tau_j\|} \leqslant \prod_j \delta^{(n(\tau_j))}$ is the upper bound for the absolute value of the product $\prod_j$ in (35). To calculate another product $\prod_i$ in (35) we use expression (33) for every factor in this product. Given γ_i the ratio in the right-hand side of (33) can be estimated as follows. In view of the definition (24) of a precontour partition function and the exponential decay condition (11) on the potential, the numerator of this ratio does not exceed

$$L((\bar{\gamma}_i\backslash\partial^{\mathrm{I}}\bar{\gamma}_i)\cap V)\, E^*(\sigma(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V))^{-1}\, E(\sigma'(\partial^{\mathrm{E}}V\cap\bar{\gamma}_i))$$
$$\times\left(\prod_{P\in\gamma_i}(2\beta e^{-n(P)})\right)$$
$$\times\sum_{\sigma((\bar{\gamma}_i\backslash\partial^{\mathrm{I}}\bar{\gamma}_i)\cap V)}\prod_{P\prec\gamma_i}(1+U(\beta,\sigma(P\cap V)+\sigma'(P\cap\hat{V}))) \tag{39}$$

while in view of the first equality in expression (27) the denominator is equal to

$$L((\bar{\gamma}_i\backslash\partial^{\mathrm{I}}\bar{\gamma}_i)\cap V)\, E(\sigma(\partial^{\mathrm{I}}\bar{\gamma}_i\cap V)+\sigma'(\partial^{\mathrm{E}}V\cap\bar{\gamma}_i))$$
$$\times\sum_{\sigma((\bar{\gamma}_i\backslash\partial^{\mathrm{I}}\bar{\gamma}_i)\cap V)}\prod_{P\subseteq(\bar{\gamma}_i\backslash\partial^{\mathrm{I}}\bar{\gamma}_i)\cap V}(1+U(\beta,\sigma(P))) \tag{40}$$

Together with the estimate

$$\prod_{\substack{P\prec\gamma_i\\ P\cap\partial^{\mathrm{E}}V\neq\varnothing}}(1+U(\beta,\sigma(P\cap V)+\sigma'(P\cap\hat{V})))$$
$$=\exp\left[\sum_{\substack{P\prec\gamma_i\\ P\cap\partial^{\mathrm{E}}V\neq\varnothing}}U(\sigma(P\cap V)+\sigma'(P\cap\hat{V}))\right]$$
$$\leqslant\exp\left[\sum_{x\in\partial^{\mathrm{E}}V\cap\bar{\gamma}_i}\sum_{P\ni x}\beta e^{-n(P)}\right]$$
$$\leqslant\exp(\beta^{1/2}\,|\partial^{\mathrm{E}}V\cap\bar{\gamma}_i|)$$
$$\leqslant\prod_{P\in\gamma_i}\exp[\beta^{1/2}n(P)] \tag{41}$$

it gives the lemma. ∎

Lemma 2 allows us to verify the general condition (A.3) of Theorem A.1 for contours $\Omega=(\{\gamma_i\},\{\tau_j\},\sigma)$.

Lemma 3. For β and δ small enough and

$$a(\Omega)=\tfrac{1}{12}\sum_i\sum_{P\in\gamma_i}n(P)+\sum_j n(\tau_j) \tag{42}$$

one has

$$\sum_{\Omega'\nsim\Omega}|W(\Omega'\mid\sigma'(\hat{V}))|\,e^{2a(\Omega')}\leqslant a(\Omega) \tag{43}$$

uniformly in V and $\sigma'(\tilde{V})$.

Proof. Consider a sublattice of $\mathbf{Z}^2$ formed by the points $\tilde{x}=(x_1, Lx_2)$ which we call basic points (x_1 and x_2 are integers). Observe that if contours Ω and Ω' are incompatible, then there exist basic points $\tilde{x}\in\tilde{\Omega}$ and $\tilde{x}'\in\tilde{\Omega}'$ which either coincide or are nearest neighbors in the sublattice of basic points $[\widetilde{\operatorname{dist}}(\tilde{x}, \tilde{x}')\leqslant 1]$. The number of basic points in $\tilde{\Omega}$ does not exceed $4\sum_i \sum_{P\in\gamma_i} n(P_i)+\sum_j n(\tau_j)$. Hence

$$
\begin{aligned}
&\sum_{\Omega' \not\sim \Omega} |W(\Omega' \mid \sigma'(\hat{V}))|\, e^{2a(\Omega')} \\
&\quad\leqslant \sum_{\tilde{x}\in\tilde{\Omega}} \sum_{\substack{\tilde{\Omega}':\exists \tilde{x}'\in\tilde{\Omega}',\\ \widetilde{\operatorname{dist}}(\tilde{x},\tilde{x}')\leqslant 1}} |W(\Omega' \mid \sigma'(\hat{V}))|\, e^{2a(\Omega')} \\
&\quad\leqslant \Big(20\sum_i \sum_{P\in\gamma_i} n(P_i)+5\sum_j \eta(\tau_j)\Big)\times \sum_{\Omega':\tilde{\Omega}'\ni 0} |W(\Omega' \mid \sigma'(\hat{V}))|\, e^{2a(\Omega')}
\end{aligned}
\tag{44}
$$

Thus, the problem is reduced to the estimation of

$$\sum_{\Omega:\tilde{\Omega}\ni 0} |W(\Omega \mid \sigma'(\hat{V}))|\, e^{2a(\Omega)}$$

We will actually estimate the even larger expression

$$\sum_{\Omega:\bar{\Omega}\ni 0} |W(\Omega \mid \sigma'(\hat{V}))|\, e^{2a(\Omega)} \tag{45}$$

The difference is that we used $\bar{\Omega}$ instead of $\tilde{\Omega}$ and we will use the upper bound of (45) in the next section.

Denote by $\omega=(\{\gamma_i\}, \{\tau_j\})$ the collection of precontours and vertical segments which can be completed to some contour $\Omega=(\{\gamma_i\}, \{\tau_j\}, \sigma)$ and set

$$W(\omega)=\prod_j \delta^{n(\tau_j)/2} \prod_i \prod_{P\in\gamma_i} 2\beta e^{-n(P)/3} \tag{46}$$

The notations $\dot{\omega}(\Omega)$, $\tilde{\omega}$, $\bar{\omega}$, $a(\omega)$, etc., are self-explanatory. For δ and β small enough the sum (45) does not exceed

$$\sum_{\omega:\bar{\omega}\ni 0} W(\omega) \sum_{\Omega:\omega(\Omega)=\omega} E\Big(\sigma\Big(\Big(\bigcup_i \partial^1\bar{\gamma}_i\Big)\cap V\Big)\Big)^{-1} E^*\Big(\sigma\Big(\Big(\bigcup_i \partial^1\bar{\gamma}_i\Big)\cap V\Big)\Big)^{-1} \tag{47}$$

Since $L \geqslant n_0$ the internal sum in (47) is equal to

$$\sum_{\sigma((\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i)\cap V)} E\left(\sigma\left(\left(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i\right)\cap V\right)\right)^{-1} E^*\left(\sigma\left(\left(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i\right)\cap V\right)\right)^{-1}$$
$$= \prod_{x\in(\bigcup_i \partial^{\mathrm{I}}\bar{\gamma}_i)\cap V} \sum_{\sigma_x} e_{\sigma_x} e^*_{\sigma_x} = 1 \tag{48}$$

To estimate the first sum in (47) we provide every ω such that $\bar{\omega} \ni 0$ with the following treelike structure. Fix $\tau_j \ni 0$ or γ_i with $\bar{\gamma}_i \ni 0$ as the root (vertex of the zeroth level) of the tree. If it is the segment τ_j, then there exists one or two precontours connected with τ_j and we choose them as the vertices of the first level of the tree. If the root is the precontour γ_i, then all segments connected with $\partial^{\mathrm{I}}\bar{\gamma}_i$ are chosen to be the vertices of the first level of the tree. This procedure can be repeated for every vertex of the first level, which gives vertices of the second level and so on until all precontours and segments of ω are selected as the vertices of the tree.

First we consider ω: $\bar{\omega} \ni 0$ with the corresponding tree being a zero-level tree (i.e., it contains the root only). If ω consists of a single segment τ and $\delta \leqslant 1/64$, then

$$\sum_{\omega:\bar{\omega}\ni 0} W(\omega) \leqslant \sum_{n(\tau)=1}^{\infty} \delta^{n(\tau)/2} \leqslant \delta^{1/3} \tag{49}$$

The case when ω consists of a single precontour γ is more complicated. Let

$$W(\gamma) = \prod_{P\in\gamma_i} 2\beta e^{-n(P)/6} \tag{50}$$

and $|\gamma|$ denotes the number of rectangles in the precontour γ. By induction in $|\gamma|$ we prove that

$$\sum_{\gamma:\bar{\gamma}\ni 0} W(\gamma) \leqslant \beta^{1/3} \tag{51}$$

For $|\gamma| = 1$, i.e., for $\gamma = \{P\}$

$$\sum_{\gamma=\{P\}:P\ni 0} 2\beta e^{-n(P)/6} = 2\beta \sum_{n=1}^{\infty} n^2 e^{-n/6} \leqslant \beta^{1/3} \tag{52}$$

if β is small enough. Suppose that for $\gamma = \{P_j\}$ with $|\gamma| < m$ it was already proven that

$$\sum_{\substack{\gamma:|\gamma|<m,\\ \bar{\gamma}\ni 0}} \prod_{P\in\gamma} 2\beta e^{-n(P')/6} \leqslant \beta^{1/3} \tag{53}$$

and consider all $\gamma=\{P_j\}$ with $|\gamma|\leqslant m$. Then

$$\begin{aligned}
&\sum_{\substack{\gamma:|\gamma|\leqslant m,\\ \bar\gamma\ni 0}} \prod_{P\in\gamma} 2\beta e^{-n(P)/6}\\
&\leqslant \sum_{P\ni 0} 2\beta e^{-n(P)/6} \prod_{x\in\partial^{E}P\cup\partial^{I}P}\Big(1+\sum_{\substack{\gamma':|\gamma'|<m,\\ \bar\gamma'\ni x}}\prod_{P'\in\gamma'} 2\beta e^{-n(P')/6}\Big)\\
&\leqslant \sum_{P\ni 0} 2\beta e^{-n(P)/6} \prod_{x\in\partial^{E}P\cup\partial^{I}P}(1+\beta^{1/3})\\
&\leqslant \sum_{P\ni 0} 2\beta e^{-n(P)/6+8\beta^{1/3}n(P)}\\
&\leqslant \sum_{P\ni 0} 2\beta e^{-(1/12)n(P)}\\
&=2\beta\sum_{n=1}^{\infty} n^2 e^{-(1/12)n}\leqslant \beta^{1/3}
\end{aligned}\tag{54}$$

where the last two inequalities hold for β small enough.

Now by induction in numver of levels in the tree corresponding to ω with $\bar\omega\ni 0$ we prove that

$$\sum_{\omega:\bar\omega\ni 0} W(\omega)\leqslant\beta^{1/3}+\delta^{1/3}\tag{55}$$

Bounds (49) and (51) verify the first step of the induction. Denote by $|\omega|$ the number of levels in ω and suppose that

$$\sum_{\substack{\omega:\bar\omega\ni 0\\ |\omega|<m}} W(\omega)\leqslant\beta^{1/3}+\delta^{1/3}\tag{56}$$

Then

$$\begin{aligned}
&\sum_{\substack{\omega:\bar\omega\ni 0,\\ |\omega|\leqslant m}} W(\omega)\\
&\leqslant \sum_{\tau:\tau\ni 0}\delta^{n(\tau)/2}\prod_{x\in a(\tau)\cup b(\tau)}\Big(1+\sum_{\substack{\omega':\bar\omega'\ni x,\\ |\omega'|<m}} W(\omega')\Big)\\
&\quad+\sum_{\gamma:\bar\gamma\ni 0}\prod_{P\in\gamma} 2\beta e^{-n(P)/3}\prod_{x:\,\mathrm{dist}\{x,P\}=1}\Big(1+\sum_{\substack{\omega':\bar\omega'\ni x,\\ |\omega'|<m}} W(\omega')\Big)\\
&\leqslant \sum_{\tau:\tau\ni 0}\delta^{n(\tau)/2} e^{(\delta^{1/3}+\beta^{1/3})\,n(\tau)}+\sum_{\gamma:\bar\gamma\ni 0}\prod_{P\in\gamma} 2\beta e^{-n(P)/3} e^{(\delta^{1/3}+\beta^{1/3})\,4n(P)}\\
&\leqslant \sum_{n(\tau)=1}^{\infty}\delta^{(5/12)\,n(\tau)}+\sum_{\gamma:\bar\gamma\ni 0} W(\gamma)\\
&\leqslant \delta^{1/2}+\beta^{1/3}
\end{aligned}\tag{57}$$

This finishes the induction and provides the upper bound $\delta^{1/3}+\beta^{1/3}$ for (45). Substituting this bound on the right-hand side of (44) gives lemma. ∎

Proof of Theorem. Applying Theorem A.1 to the representation (36), we obtain an absolutely convergent polymer expansion for $\log \Xi(V \mid \sigma'(\hat{V}))$. ∎

5. UNIQUENESS OF GIBBS STATES AND EXPONENTIAL DECAY OF CORRELATIONS

Lemma 4. The limit Gibbs distribution for the Hamiltonian (18) does not depend on boundary conditions.

Proof. First we recall that for a given sequence of configurations $\sigma'_m(\mathbf{Z}^2)$ the corresponding limit Gibbs distribution is the measure on $\Sigma_M^{\mathbf{Z}}$ with the marginal distributions

$$\mu_{\{\sigma'_m\}}(\sigma(A)) = \lim_{m\to\infty} \sum_{\tilde{\sigma}(V_m\setminus A)} \mu_{V_m,\sigma'_m}(\sigma(A)+\tilde{\sigma}(V_m\setminus A)) \tag{58}$$

where $A \subset \mathbf{Z}^2$ is finite, $\sigma(A)$ is a configuration in A (local observable), and V_m is a sequence of volumes growing to $\mathbf{Z}^2$ in the van Hove sense. By the consistency of the Gibbs distribution we may assume that A is a rectangle of size $l(A)\times Ll(A)$.

It is not hard to see now that for $V=V_m$ and $\sigma'=\sigma'_m$ the sum on the right-hand side of (58) is equal to

$$\begin{aligned}
&\frac{L(A)}{E^*(\sigma(\partial^1 A))}\exp\left[\beta\sum_{P\subseteq A} U(\sigma(P))\right]\frac{\Xi(V\setminus A \mid \sigma'(\hat{V})+\sigma(A))}{\Xi(V\mid\sigma'(\hat{V}))}\\
&\quad=\exp\left[\beta\sum_{P\subseteq A}U(\sigma(P))+\sum_{\pi:\pi\cap V\setminus A\neq\varnothing}W(\pi\mid\sigma(A)+\sigma'(\hat{V}))\right.\\
&\qquad\left.-\sum_{\pi:\pi\cap V\neq\varnothing}W(\pi\mid\sigma'(\hat{V}))\right]
\end{aligned}\tag{59}$$

where for $\pi=[\Omega_i^{\alpha_i}]$ we denote $\bar{\pi}=\bigcup_i\bar{\Omega}_i$.

Note that $W(\pi\mid\sigma(A)+\sigma'(\hat{V}))=W(\pi\mid\sigma'(\hat{V}))$ if $\operatorname{dist}(\bar{\pi},A)>1$, $W(\pi\mid\sigma(A)+\sigma'(\hat{V}))=W(\pi\mid\sigma(A))$ if $\operatorname{dist}(\bar{\pi},\hat{V})>1$, and $W(\pi\mid\sigma'(\hat{V}))=W(\pi)$ if $\operatorname{dist}(\bar{\pi},\hat{V})>1$. Thus we have

$$\sum_{\pi:\pi\cap V\backslash A\neq\varnothing} W(\pi\mid\sigma(A)+\sigma'(\hat{V}))-\sum_{\pi:\pi\cap V\neq\varnothing} W(\pi\mid\sigma'(\hat{V}))$$

$$=\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{A})=0}} W(\pi\mid\sigma(A))-\sum_{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1} W(\pi)$$

$$+\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{V})=1}} W(\pi\mid\sigma(A)+\sigma'(\hat{V}))-\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{V})=1}} W(\pi\mid\sigma'(\hat{V}))$$

$$-\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{V})=1}} W(\pi\mid\sigma(A))+\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{V})=1}} W(\pi)$$

When $V\to\mathbf{Z}^2$ each of the last four sums in the above expression is less than

$$5\,|A|\exp[-\tfrac{1}{12}\mathrm{dist}(A,\hat{V})] \tag{60}$$

uniformly in $\sigma'(\hat{V})$. In fact, according to Lemma 3,

$$w(\Omega)=W(\Omega)\,e^{a(\Omega)} \tag{61}$$

satisfies condition (A.3) of Theorem A.1. So for the corresponding $w(\Omega)$ statistical weights of polymers $w(\pi)$ one has the estimate (A.6). Now

$$\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{V})=1}} |W(\pi)|\leqslant\sum_{x:\,\mathrm{dist}(x,A)\leqslant 1}\;\sum_{\Omega:\,\pi\ni x}\;\sum_{\substack{\pi:\,\pi\ni\Omega,\\ \mathrm{dist}(\bar{\pi},\hat{V})=1}} |W(\pi)|$$

$$\leqslant\exp[-\tfrac{1}{12}\mathrm{dist}(A,\hat{V})]\sum_{x:\,\mathrm{dist}(x,A)\leqslant 1}\;\sum_{\Omega:\,\bar{\Omega}\ni x}\;\sum_{\pi:\,\pi\ni\Omega} w(\pi)$$

$$\leqslant\exp[-\tfrac{1}{12}\mathrm{dist}(A,\hat{V})]\sum_{x:\,\mathrm{dist}(x,A)\leqslant 1}\;\sum_{\Omega:\,\bar{\Omega}\ni x} w(\Omega)\,e^{a(\Omega)}$$

$$\leqslant\exp[-\tfrac{1}{12}\mathrm{dist}(A,\hat{V})]\sum_{x:\,\mathrm{dist}(x,A)\leqslant 1}(\beta^{1/3}+\delta^{1/3})$$

$$\leqslant 5\,|A|\exp[-\tfrac{1}{12}\mathrm{dist}(A,\hat{V})] \tag{62}$$

where we used (A.6). The estimation of the remaining sums over $W(\pi\mid\cdot)$ can be performed in the same way. This gives

$$\mu_{\{\sigma'_m\}}(\sigma(A))$$

$$=\frac{L(A)}{E^*(\sigma(\partial^1 A))}$$

$$\times\exp\left[\beta\sum_{P\subseteq A}U(\sigma(P))+\sum_{\substack{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1\\ \mathrm{dist}(\bar{\pi},\hat{A})=0}} W(\pi\mid\sigma(A))-\sum_{\pi:\,\mathrm{dist}(\bar{\pi},A)\leqslant 1} W(\pi)\right] \tag{63}$$

where the second sum is taken over polymers of $\mathbf{Z}^2\backslash A$ and the third sum is taken over free polymers of $\mathbf{Z}^2$. This measure is obviously independent of the boundary conditions $\{\sigma'_m\}$. ▮

Lemma 5. Let $\mu(\cdot)$ denotes the unique Gibbs state from Lemma 4. Then for any finite $A, B \subset \mathbf{Z}^2$ with

$$\min(|A|, |B|) \exp[-(1/12) \operatorname{dist}(A, B)] \leqslant 1/4$$

and any configurations $\sigma(A)$ and $\sigma(B)$

$$\begin{aligned} &|\mu(\sigma(A)+\sigma(B)) - \mu(\sigma(A))\, \mu(\sigma(B))| \\ &\quad \leqslant 10\mu(\sigma(A))\, \mu(\sigma(B)) \min(|A|, |B|)\, e^{-(1/12)\operatorname{dist}(A,B)} \end{aligned} \tag{64}$$

Proof. We may again assume that both A and B are appropriate rectangles. The proof is similar to the proof of the previous lemma and it can be derived from the following representation of the difference on the left-hand side of (64):

$$\begin{aligned} &\mu(\sigma(A))\, \mu(\sigma(B)) \left\{ \exp \left[\sum_{\substack{\pi:\, \operatorname{dist}(\bar{\pi}, A \cup B) \leqslant 1 \\ \operatorname{dist}(\bar{\pi}, \widehat{A \cup B}) = 0}} W(\pi \mid \sigma(A)+\sigma(B)) \right.\right. \\ &\quad - \sum_{\pi:\, \operatorname{dist}(\bar{\pi}, A \cup B) \leqslant 1} W(\pi) - \sum_{\substack{\pi:\, \operatorname{dist}(\bar{\pi}, A) \leqslant 1 \\ \operatorname{dist}(\bar{\pi}, \hat{A}) = 0}} W(\pi \mid \sigma(A)) + \sum_{\pi:\, \operatorname{dist}(\bar{\pi}, A) \leqslant 1} W(\pi) \\ &\quad \left.\left. - \sum_{\substack{\pi:\, \operatorname{dist}(\bar{\pi}, B) = 1 \\ \operatorname{dist}(\bar{\pi}, \hat{B}) = 0}} W(\pi \mid \sigma(B)) + \sum_{\pi:\, \operatorname{dist}(\bar{\pi}, B) \leqslant 1} W(\pi) \right] - 1 \right\} \\ &= \mu(\sigma(A))\, \mu(\sigma(B)) \left\{ \exp \left[- \sum_{\substack{\pi:\, \operatorname{dist}(\bar{\pi}, A) \leqslant 1 \\ \operatorname{dist}(\bar{\pi}, B \leqslant 1}} W(\pi \mid \sigma(A)+\sigma(B)) \right.\right. \\ &\quad \left.\left. + \sum_{\substack{\pi:\, \operatorname{dist}(\bar{\pi}, A) \leqslant 1 \\ \operatorname{dist}(\bar{\pi}, B) \leqslant 1}} W(\pi) \right] - 1 \right\} \end{aligned} \tag{65}$$

which gives (64) for $\min(|A|, |B|) \exp[-\frac{1}{6} \operatorname{dist}(A, B)] \ll 1$. ▮

6. GENERALIZATIONS

From the point of view of one-dimensional statistical mechanics a subshift of finite type is a one-dimensional model given by the nearest neighbor potential taking values 0 and $-\infty$. The corresponding transfer matrix M has only 0 and 1 entries and the number of configurations

$N(V_n \mid \sigma'_{a(V_n)}, \sigma'_{b(V_n)})$ is nothing but the partition function in the volume V_n with the boundary conditions $\sigma'_{a(V_n)}$ and $\sigma'_{b(V_n)}$.

Clearly the representation (13) for the partition function and the estimate (14) are still true for any one-dimensional model on the configuration space $S^{\mathbf{Z}}$ with nonnegative transfer-matrix having positive power. This extends our results to the case when on the RHS of (2) one has an additional term

$$- \sum_{I \cap V \neq \varnothing,\, |I| = 2} U(\sigma(I \cap V) + \sigma'(I \cap \hat{V})) \tag{66}$$

where I denotes a vertical segment and $U(\sigma(I)) \in \mathbf{R} \cup \{\infty\}$.

The extension to the case of arbitrary finite-range vertical potential is also standard. Suppose that instead of (66) we have

$$- \sum_{I \cap V \neq \varnothing,\, |I| \leqslant m} U(\sigma(I \cap V) + \sigma'(I \cap \hat{V})) \tag{67}$$

where $m > 2$. Partition the lattice $\mathbf{Z}^2$ into vertical segments of length $2m$ such that the beginning of every segment (i.e., its lower site) belongs to the sublattice $\bar{\mathbf{Z}}^2 = \mathbf{Z} \times 2m\mathbf{Z}$. Now we pass from the spin space S and lattice $\mathbf{Z}^2$ to the spin space $\bar{S} = S^{2m}$ and lattice $\bar{\mathbf{Z}}^2$. Namely, for vertical segments $\bar{I} = \bar{I}_1 \cup \bar{I}_2$ of $\bar{\mathbf{Z}}^2$ with $|\bar{I}| = 2$, $|\bar{I}_1| = 1$, $|\bar{I}_2| = 1$ ($|\cdot|$ in units of $\bar{\mathbf{Z}}^2$) we set

$$\bar{U}(\bar{\sigma}(\bar{I})) = \tfrac{1}{2} \sum_{I \subseteq \bar{I}_1,\, I \subseteq \bar{I}_2} U(\sigma(I)) + \sum_{I \cap \bar{I}_1 \neq \varnothing,\, I \cap \bar{I}_2 = \varnothing} U(\sigma(I)) \tag{68}$$

and for squares $\bar{Q}$ of $\bar{\mathbf{Z}}^2$ (which are rectangles of $\mathbf{Z}^2$) we set

$$\bar{U}(\bar{\sigma}(\bar{Q})) = \sum_{Q: b(Q) \sim b(\bar{Q})} U(\sigma(Q)) \tag{69}$$

where by $\sim$ we refer to the natural relation between initial and "bared" objects that was discribed in Section 3. By definition, $\bar{U}(\bar{\sigma}(\bar{I}))$ is not 0 only for $|\bar{I}| = 2$ and clearly $\bar{U}(\bar{\sigma}(\bar{Q})) \leqslant 2m \exp[-n(\bar{Q})]$, which reduces this case to the previous one by absorbing $2m$ in β.

The situation is slightly more involved for the cse of infinite-range exponentially decaying vertical potential. In that case an additional vertical term is

$$- \sum_{I \cap V \neq \varnothing} U(\sigma(I \cap V) + \sigma'(I \cap \hat{V})) \tag{70}$$

with no restrictions on $|I|$ and $|U(\sigma(I))| \leqslant C \exp(-\alpha\, |I|)$, where $C > 0$ and $\alpha > 0$. Now we cut the infinite-range potential at some range m, the value of which will be chosen later, and transform the resulting finite-range potential into a nearest neighbor one as in the previous paragraph. It is a well-known fact that the resulting v and ρ in (13) can be chosen independent of m (for a short proof see, e.g., Appendix in ref. 1). During the transformation of the model we also arrange the tail of the infinite-range vertical potential into a square potential $\bar{U}'(\bar{s}(\bar{Q}))$ in the following way:

$$\bar{U}'(\bar{s}(\bar{Q})) = \sum_{\substack{I:\, I \subset \bar{Q},\, I \not\subset \bar{Q}' \\ \forall \bar{Q}' \subset \bar{Q}}} U(s(I)) \tag{71}$$

Clearly

$$|\bar{U}'(\bar{s}(\bar{Q}))| \leqslant mCn(\bar{Q}) \exp[-(m-1)\,\alpha n(\bar{Q})] \tag{72}$$

It is not hard to see that the results of previous sections remains true if (1) is replaced by

$$|U(\sigma(Q))| \leqslant \exp[-\alpha l(Q)], \qquad 0 < \alpha < 1 \tag{73}$$

The affected estimates are (6), (20), (41), (52), and (54) and the corresponding modifications are straightforward. Given v, ρ, and α, we take now L and then β_0 such that for $\beta \leqslant \beta_0$ and (73) instead of (1) the Main Result is true. Choosing m so large and then β so small that

$$mCn(\bar{Q}) \exp[-(m-1)\,\alpha n(\bar{Q})] \leqslant \frac{\beta_0}{2} \exp[-\alpha n(\bar{Q})], \qquad 2m\beta \leqslant \frac{\beta_0}{2}$$

we obtain

$$|\beta \bar{U}(\bar{s}(\bar{Q})) + \bar{U}'(\bar{s}(\bar{Q}))| \leqslant \beta_0 \exp[-\alpha n(\bar{Q})]$$

which extends the Main Result to the case of the infinite-range vertical potential.

APPENDIX. POLYMER EXPANSION THEOREM

Consider a finite or countable set Θ the elements of which are called (abstract) *contours* and denoted θ, θ', etc. Fix some reflexive and symmetric relation on $\Theta \times \Theta$. A pair $\theta, \theta' \in \Theta \times \Theta$ is called incompatible ($\theta \not\sim \theta'$) if it belongs to a given relation and this pair is called compatible ($\theta \sim \theta'$) in the opposite case. A collection $\{\theta_i\}$ is called a *compatible collection of contours* if any two of its elements are compatible. Every contour θ is assigned a (generally speaking) complex-valued *statistical weight* denoted by $w(\theta)$, and for any finite $\Lambda \subseteq \Theta$ an (abstract) *partition function* is defined as

$$Z(\Lambda) = \sum_{\{\theta_j\} \subseteq \Lambda} \prod_j w(\theta_j) \tag{A.1}$$

where the sum is extended to all compatible collections of contours $\theta_i \in \Lambda$. The empty collection is compatible by definition, and it is included in $Z(\Lambda)$ with statistical weight 1.

A *polymer* $\pi = [\theta_i^{\alpha_i}]$ is an (unordered) finite collection of different contours $\theta_i \in \Theta$ taken with positive integer multiplicities α_i, such that for every pair θ', $\theta'' \in \pi$ there exists a sequence $\theta' = \theta_{i_1}, \theta_{i_2}, ..., \theta_{i_s} = \theta'' \in \pi$ with $\theta_{i_j} \nsim \theta_{i_{j+1}}$, $j = 1, 2, ..., s-1$. The notation $\pi \subseteq \Lambda$ means that $\theta_i \in \Lambda$ for every $\theta_i \in \pi$.

With every polymer π we associate an (abstract) graph $\Gamma(\pi)$ which consists of $\sum_i \alpha_i$ vertices labeled by the contours from π and edges joining every two vertices labeled by incompatible contours. It follows from the definition of $\Gamma(\pi)$ that it is connected and we denote by $r(\pi)$ the quantity

$$r(\pi) = \prod_i (\alpha_i!)^{-1} \sum_{\Gamma' \subset \Gamma(\pi)} (-1)^{|\Gamma'|} \tag{A.2}$$

where the sum is taken over all connected subgraphs Γ' of $\Gamma(\pi)$ containing all of $\sum_i \alpha_i$ vertices and $|\Gamma'|$ denotes the number of edges in Γ'. For any $\theta \in \pi$ we denote by $\alpha(\theta, \pi)$ the multiplicity of θ in the polymer π.

The following polymer expansion theorem is a modification of results of refs. 16 and 12 proven in ref. 14 (see also ref. 5 for close results).

Theorem A.1. Suppose that there exists a function $a(\theta)$: $\Theta \mapsto \mathbf{R}^+$ such that for any contour θ

$$\sum_{\theta' : \theta' \nsim \theta} |w(\theta')| \, e^{a(\theta')} \leqslant a(\theta) \tag{A.3}$$

Then, for any finite Λ,

$$\log Z(\Lambda) = \sum_{\pi \subseteq \Lambda} w(\pi) \tag{A.4}$$

where the statistical weight of a polymer $\pi = [\theta_i^{\alpha_i}]$ is equal to

$$w(\tau) = r(\pi) \prod_i w(\theta_i)^{\alpha_i} \tag{A.5}$$

Moreover, the series (A.4) for $\log Z(\Lambda)$ is absolutely convergent in view of the estimate

$$\sum_{\pi:\pi\ni\theta} \alpha(\theta, \pi)\, |w(\pi)| \leqslant |w(\theta)|\, e^{a(\theta)} \tag{A.6}$$

which holds true for any contour θ.

Corollary A.2. For any polymer $\pi = [\theta_i^{\alpha_i}]$

$$|r(\pi)| \leqslant \min_{\theta_i\in\pi}(\alpha(\theta_i, \pi)^{-1}\, |w(\theta_i)|\, e^{a(\theta_i)}) \prod_i |w(\theta_i)|^{-\alpha_i} \tag{A.7}$$

Proof. Denote by θ^* the contour from π such that

$$\alpha(\theta^*, \pi)^{-1}\, |w(\theta^*)|\, e^{a(\theta^*)} = \min_{\theta_i\in\pi}(\alpha(\theta_i, \pi)^{-1}\, |w(\theta_i)|\, e^{a(\theta_i)}) \tag{A.8}$$

According to (A.6),

$$\alpha(\theta^*, \pi)\, |w(\pi)| \leqslant \sum_{\pi':\pi'\ni\theta^*} \alpha(\theta^*, \pi')\, |w(\pi')| \leqslant |w(\theta^*)|\, e^{a(\theta^*)} \tag{A.9}$$

and (A.7) follows now from definition (A.5). ∎

Corollary A.3. For any function $b(\theta)$: $\Theta \mapsto \mathbf{R}^+$ consider modified statistical weights of contours $\tilde{w}(\theta)$ such that

$$|\tilde{w}(\theta)| = |w(\theta)|\, e^{-b(\theta)} \tag{A.10}$$

Then for the corresponding statistical weights of polymers $\tilde{w}(\pi)$ one has the bound

$$|\tilde{w}(\pi)| \leqslant \min_{\theta_i\in\pi}(\alpha(\theta_i, \pi)^{-1}\, |w(\theta_i)|\, e^{a(\theta_i)}) \exp\left[-\sum_i \alpha_i b(\theta_i)\right] \tag{A.11}$$

Proof. Substituting (A.7) into the definition of $\tilde{w}(\pi)$, one immediately gets (A.11). ∎

Suppose that, given a compatibility relation for contours, one can find the maximal statistical weight $w(\theta)$ satisfying (A.3) with some $a(\theta)$. Then Corollary A.3 says that for the statistical weights $\tilde{w}(\theta)$ of contours that are smaller in absolute value, the corresponding statistical weights of polymers decay exponentially.

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